Investigation of infinite series

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The concept of infinity has been known to mankind for thousands of years, but it wasn't until mathematics came along that people started experimenting with it and finding out information about it. One of the first contributions to this topic was made by Ancient Greek mathematicians and philosophers, such as Archimedes and Zeno. So let us start looking at infinite series by investigating the problems created by the Ancient Greeks. Zeno was a famous philosopher of Ancient Greece, who lived in the fifth century BC and his paradoxes have challenged the views of great philosophers, but even though the philosophical debate that arises from these paradoxes is a very intriguing topic, we will only consider the mathematics of these paradoxes and solutions to these paradoxes.

Probably the most famous and interesting mathematically is the paradox concerning Achilles and the turtle. Aristotle puts this problem as follows [1]:

> "The second is the so-called 'Achilles', and it amounts to this, that in a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

So the problem is that, if you propose the question like that, from the first glance it seems that Achilles will never reach the tortoise, thus creating the paradox. But as we will now see, this is not the case due to mathematics.

To simplify the problem let us assume that Achilles decides to give the tortoise a head-start of 100 meters and is 10 times faster than the tortoise, and both the tortoise and Achilles do not accelerate or decelerate during the race. So when Achilles reaches the starting position of the turtle, the turtle would have advanced another 10 meters. Then when Achilles reaches that position, the turtle would have advanced another meter and so on. This struggle of Achilles trying to reach the tortoise is illustrated in Figure 1[2]. At first it seems that Achilles indeed never reaches the tortoise as we are adding infinitely many distances together, so surely the total distance must be infinite. But this is not the case. Let us denote the distance traveled in each interval by Achilles by S_n and the total distance between the start and the point at which Achilles overtakes the turtle by S.

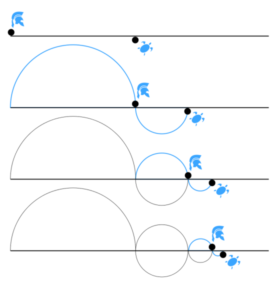

Figure 1: The race between Achilles and the tortoise

Then we can say $S_1 = 100$, $S_2 = 10$ and so on. From this it is easy to see that

$$S_n = 10^{3-n}$$

Now we see that as n gets larger and larger S_n gets smaller and smaller or to put it mathematically $S_n \to 0$ as $n \to \infty$. This means that actually the distance needed for Achilles to catch up to the tortoise gets smaller and smaller and can be made infinitesimally small by taking n large enough.

This eliminates the paradox, but now the question is how far does Achilles needs to travel to overtake the tortoise? The distance is quite easily calculated as:

$$100 + 10 + 1 + 0.1 + 0.01 + ... = 111.\dot{1} = 111 + \frac{1}{9}$$

and so Achilles will be already ahead of the tortoise after 112 meters thus solving the paradox. After all there is no paradox, but this was a very important problem for mathematics as it was one of the first problems to show that you can sum infinitely many numbers, but get a finite answer.

But before we look at any other infinite series in particular, let us first define what an infinite series is. An infinite series of a give sequence (a_n) is defined as the limit of the sequence (S_n), given by :

$$S_n = a_1 + a_2 + a_3 + ... + a_n$$

or in mathematical notation:

$$\sum_{n=1}^{\infty} a_n = \lim_{x\to\infty} S_n$$

If the limit exists than we call the series convergent, otherwise the series is called divergent.

So let us start by what can be considered as the simplest infinite series, the series of natural numbers. This series was studied by many great mathematicians including Euler, Ramanujan and others and the answer which they got was rather interesting. So let us look at these sequence, which mathematically is simply written as:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ...$$

It is obvious just from looking at the series that this is series is divergent, i.e. it doesn't have a finite sum. But what Euler and Ramanujann discovered, is the fact that the series can be manipulated in such a way as to give a finite sum. This is what Ramanujan has written in one of his first notebooks[3]:

$$c \quad = 1 + 2 + 3 + 4 + ... \tag{1}$$

$$\therefore \quad 4c \quad = \quad 4 \quad + \quad 8 + ... \tag{2}$$

$$\therefore \quad -3c \quad = 1 - 2 + 3 - 4 + ... = \frac{1}{(1+1)^2} = \frac{1}{4} \tag{3}$$

$$\therefore \quad c \quad = -\frac{1}{12} \tag{4}$$

So as we can see that Ramanujan got a finite result for the infinite series of natural numbers and even more surprising is the fact the final result is a negative fraction even though we were only adding natural numbers.

Of course the method which Ramanujan used in this proof is flawed and in fact this proof is flawed in two places. The first flaw is the fact that one cannot manipulate infinite series in the same way as finite series. Here we see an example that appending finitely or infinitely many zeros can create different results for the same series. This problem shows that this method of summation isn't linear or stable, which basically means that if you add another 0 to both sides, the value you get will be different for each 0 you add. And the second problem with this proof is in line 3 and it is the fact that Ramanujan used the formula for the sum of the geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

which is by definition defined for $|r| < 1$, but in this case $r = -1$. To avoid problems like these with other infinite series, mathematicians have derived several ways to distinguish between series which have a finite sum and those which don't.

There are many ways to test for convergence of the series, and below are some of those tests. Let us first consider series which only contain positive terms. But before we do this let us start with proving a theorem, which is useful as pretest for figuring out if a series is convergent. Now the theorem states that, if

$$\sum_{n=1}^{\infty} a_n \text{ is a convergent series, then } (a_n) \to 0$$

and the proof of this is quite simple and presented below.

Proof:

Let us denote the sequence of partial sums by (S_n), and note that

$$a_n = S_n - S_{n-1} \quad (n \geq 2)$$

Now since we are dealing with a convergent series, $(S_n) \to S$, where S is the value of the series, and so $a_n \to S - S = 0$ as $n \to \infty$.

□

The converse of this theorem is sadly not true and to see that we can just look at the harmonic series. The harmonic series is the sum of the reciprocals of natural numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

It is trivial that $1/n \to 0$ as $n \to \infty$, but as proven by Nicole Oresme the series is actually divergent [4].This is not the only proof of the divergence of the harmonic series, but it is the one which requires the least mathematical knowledge and it is also the first one, and considered as the pinnicale of medieval mathematics.Firstly Oresme considered the sum up to and including the 2^nth term.

$$S_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2^n - 1} + \frac{1}{2^n}$$

Then he grouped the terms as follows:

$$S_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + ... + \frac{1}{2^n}\right)$$

and notices that each bracket is greater than a half as each bracket containing $1/2^k$ contains $1/2^{k-1}$ terms and so

$$\left(\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+2} + ... + \frac{1}{2^k}\right) > \left(\frac{1}{2^k} + \frac{1}{2^k} + ... + \frac{1}{2^k}\right) = \frac{2^{k-1}}{2^k} = \frac{1}{2} \tag{5}$$

$$\therefore S_{2^n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2} \tag{6}$$

And from (6) it follows that $S_{2^n} \to \infty$ as $n \to \infty$, which implies that the harmonic series diverges.

Now let us look at the actual ways, in which one can test for convergence.The first test for convergence is the comparison test.

Let $\sum_{n=1}^{\infty} a_n$ be a known convergent series, then $\sum_{n=1}^{\infty} b_n$ is a convergent series, if $b_n \leq a_n$ for all $n \geq 1$. And the same goes for divergent series, i.e. if $\sum_{n=1}^{\infty} a_n$ be a known divergent series, then $\sum_{n=1}^{\infty} b_n$ is also divergent, if $b_n \geq a_n$ for all $n \geq 1$.

So let us prove the convergence result as the divergent one can be proven similarly:

$$\text{Let } \sum_{n=1}^{\infty} a_n = A, \text{ then the partial sum sequence } A_n \to A \text{ as } n \to \infty$$

Now let is denote the sequence of partial sums of b_n as B_n. Then $B_n \leq A_n$ as $b_n \leq a_n$. So this means that for all $n \in \mathbb{N}$, $B_n \leq A$. That means that the sequence (B_n) is bounded above and as it is an increasing sequence, we can safely say that the sequence (B_n) has the limit B which is less than or equal to A.

□

The comparison test is a powerful tool to deduce which series are convergent and which are divergent and this will be demonstrated by the following example:

Let us look at the infinite series defined by:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

To determine if this series is convergent or divergent we first notice that :

$$\frac{1}{n^2} < \frac{1}{n(n-1)} \text{ for all } n > 1$$

and now if we look at the series of the later we notice that:

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n} = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ... = 1$$

which shows that the series is convergent. And therefore by the comparison test the original series is also convergent. Here it is good to know that even though the second series start from $n = 2$, the comparison test still works as we could define $a_1 = 1$ and thus insuring that all $b_n \leq a_n$.

This is also a great example to show the problem with most tests and that is that they do not give a way of finding the value of the series. The value of this series in particular is very interesting and the problem of finding it, was known as the Basel problem. This problem was soled by the great mathematician Euler in 1734. But before we can recreate his solution, we need to look at some interesting infinite series, which come from the work of Taylor and Mclaurin.

Taylor theorem states that if a function f is n times differentiable in $[a, b]$, then there exists c in $[a, b]$ such that:

$$f(b) = f(a) + (b-a)\frac{df}{dx}(a) + \frac{(b-a)^2}{2!} \cdot \frac{d^2 f}{dx^2}(a) + ... + \frac{(b-a)^n}{n!} \cdot \frac{d^n f}{dx^n}(c)$$

The proof of the theorem isn't too complicated, but is irrelevant to us at the moment so we will continue without it. By modifying this theorem so that $a = 0$ and $b = x$ we get the version of this theorem, which is called Maclaurin's theorem:

$$f(x) = f(0) + x\frac{df}{dx}(0) + ... + \frac{x^{n-1}}{(n-1)!} \cdot \frac{d^{n-1} f}{dx^{n-1}}(0) + R_n \ ,$$

where R_n is what is called a remainder term, which has the value of:

$$\frac{x^n}{n!} \cdot \frac{d^n f}{dx^n}(\theta x) \ ,$$

where $0 < \theta < 1$.

Now it is clear that for some functions $R_n \to 0$ as $n \to \infty$ and if this is the case, we can obtain a series, which is called the Taylor-Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \frac{d^n f}{dx^n}(0)$$

and the functions which satisfy this condition are called analytic. To give an example, let us look at the sine function. We know that:

$$\frac{d(sinx)}{dx} = cosx \quad \& \quad \frac{d(cosx)}{dx} = -sinx$$

From this it easy to see that the derivatives of $sin(x)$ at 0 alternate between $0, 1, 0, -1$ in that order. Now the R_n of the sine function then is either 0 or $\pm x^n/n!$. Now as we will see later $x^n/n! \to 0$ as $n \to \infty$ and from this we can deduce that $R_n \to 0$ as $n \to \infty$. This means that sine is an analytic function and has a Taylor-Maclaurin series associated with it. Now substituting sine and it's derivatives into Maclaurin's theorem gives us:

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - ... = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Now let us return to the Basel problem. So what Euler did first was take the Maclaurin series for $sin(x)$ and instead of x he substituted πx, so he got:

$$sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \frac{(\pi x)^9}{9!} - ...$$

Then came the brilliance of Euler's solution. What he did is take note that a polynomial function $P(x)$ with the maximum power of x being n and the roots of the function being in points $a_1, a_2, a_3, ..., a_n$ can be written as:

$$P(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right) ... \left(1 - \frac{x}{a_n}\right)$$

then Euler assumed that the same can be said for a polynomial function with infinite power:

$$P(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right) ...$$

After that he noted that the roots of the function $sin\pi x$ are $0, \pm 1, \pm 2, \pm 3$ and so on, therefore the function $sin\pi x$ can be written as:

$$sin(\pi x) = \pi x\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{1}\right)\left(1 - \frac{2}{x}\right)\left(1 + \frac{2}{x}\right) ... ,$$

which can be further simplified by multiplying the corresponding brackets:

$$sin(\pi x) = \pi x(1 - x^2)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right) ...$$

Now if we expand the brackets term by term, we get:

$$sin(\pi x) = \pi x - \pi x^3\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...\right) + \pi x^5\left(\frac{1}{1 \cdot 4} + \frac{1}{1 \cdot 9} + ... + \frac{1}{4 \cdot 9} + ...\right) - ...$$

and finally if we compare the x^3 terms in the Taylor-Maclaurin series and this equation we find that

$$\frac{\pi^3}{3!} = \pi\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...\right)$$

$$\therefore 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \frac{\pi^2}{6}$$

and so we have found the value of the series. But even though the value is correct, there is a slight problem with the proof. The problem is that it is actually not that obvious, that you can factor $sinx$ into linear terms. Now this problem was solved by Weierstrass, who proved that you can indeed factor $sinx$ with what is now known as Weierstrass factorization theorem.

Now let us continue with other different methods to test a series for convergence. The next test is called the ratio test and here is how it is stated:

Let $\sum_{n=1}^{\infty} a_n$ be the series, whose convergence is to be established, then if $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) < 1$, the series is convergent and if $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) > 1$, the series is divergent. The latter includes the case where the ratio tends towards infinity.The proof of this test is quite simple and explained below.

Let us start by firstly assuming that

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = l < 1$$

Now as $l < 1$ we may choose a real number r such that $l < r < 1$

$$\therefore \quad \frac{a_{n+1}}{a_n} < r \tag{7}$$

$$\therefore \quad a_{n+1} < a_n r \tag{8}$$

$$\therefore \quad a_{n+3} < a_{n+2} r < a_{n+1} r^2 < a_n r^3 \tag{9}$$

and from (9) we can say that in general

$$a_n < a_{N+1} r^{n-(N+1)} \text{ for all } n > N \text{ ,where } n, N \in \mathbb{N}$$

so now if we compare our original series from the $(N+1)$th with the geometric series

$$\sum_{N+1}^{\infty} a_{N+1} r^{n-N-1} \ ,$$

which is clearly convergent as the common ratio r is between 0 and 1 as established earlier. Now as there are finitely many natural numbers between 1 and $N+1$, the original series is also convergent.

Now let us assume that

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = l > 1$$

Now this will mean that there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\frac{a_{n+1}}{a_n} > 1 \therefore a_{n+1} > a_n$$

and so from the fact that if a series is convergent $(a_n) \to 0$ we safely deduce that the series we are looking at is divergent.

□

There is a clear downside with using the ratio test and that is that if the limit equals to 1 then the test is inconclusive and other methods of investigating convergence have to be used. The ratio test is most commonly used in series ,which involve factorials as will be seen in our next example.

This example is taken from our work on the Taylor-Maclaurin expansion earlier and it is the following series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So let us apply the ratio test to this series.

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} = \frac{x^{n+1} n!}{(n+1)! x^n} = \frac{x}{n+1}$$

Now because

$$\frac{x}{n+1} \to 0 \text{ as } n \to \infty$$

we can conclude that this series converges for all $x \in \mathbb{R}$.

Another way and perhaps one of the most powerful ways to test for convergence of a series, is the integral test. But before we can deal with this method we have to become familiar with the concept of improper integrals of the first kind.

Now such an integral is called improper, because one or both of its limits contain infinity. If we have a function $f(x)$ defined on the interval $[a, \infty)$ than one of the possibilities for it's improper integral is

$$\int_a^\infty f(x)dx = \lim_{K\to\infty} \int_a^K f(x)dx \ .$$

Now if such an integral exists it is called convergent and if it doesn't, which in most cases means it tends to infinity, the integral is called divergent.

The integral test states that if we let $f(x)$ be a positive, decreasing function defined on the interval $[1, \infty)$, then the series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent.

To prove this fist we note that given a natural number k,

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k) \ ,$$

which can be easily illustrated by Figure 2 [5], where the green represents the lower sum and purple the higher one, while the red line is the function $f(x)$. From this equation we can deduce that

$$f(2)+f(3)+f(4)+f(5)+...+f(k) \leq \int_1^k f(x)dx \leq f(1)+f(2)+f(3)+f(4)+...+f(k-1)$$

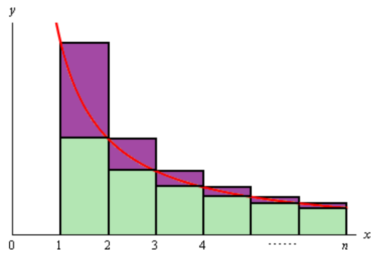

Figure 2: Illustration of the concept of the integral test

Now if we let $k \to \infty$ we get

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^\infty f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

and from this we can see that if the integral is convergent and has a value L, then the sum from 2 to ∞ will be less than or equal to L, and thus also converge. If the sum from 1 to ∞ is convergent then the integral will be less than or equal to that sum and thus also convergent. Lastly it is quite clear to see that a very similar argument can be applied for the case when the sum and the integral are divergent.

□

As said before the integral test is a very useful test as most generalized series are easier to prove convergent or divergent using the integral test. To see this consider the following example.

Let us find out for which values of s the series converges and for which it diverges, the series being

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

To do this consider the following integral:

$$\int_1^{\infty} \frac{dx}{x^s}$$

Now if $s \neq 1$ then we have the following

$$\int_1^K \frac{dx}{x^s} = \int_1^K x^{-s} dx = \left[\frac{x^{-s+1}}{-s+1}\right]_1^K = \frac{K^{1-s} - 1}{1-s} = \frac{1 - K^{1-s}}{s-1}$$

Now it is clear that $1/(s-1)$ is just a constant and so we only need to consider what happens to $1 - K^{1-s}$ as s changes. If $s > 1$ then $1 - s < 0$ and so $K^{1-s} \to 0$ as $K \to \infty$. Therefore the integral converges to 1. Now if $s < 1$ then $1 - s > 0$ and so $K^{1-s} \to \infty$ as $K \to \infty$. In this case it is clear that the integral diverges.

So the only case left to consider is the $s = 1$ case. Here

$$\int_1^K \frac{dx}{x^s} = \int_1^K \frac{dx}{x} = \Big[ln(x)\Big]_1^K = ln(K)$$

and so we can easily see that the integral diverges, because $ln(K) \to \infty$ as $K \to \infty$. From these results we can deduce the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ converges if } s > 1 \text{ and diverges if } s \leq 1.$$

So with this new knowledge let us move on to series, which contain both positive and negative numbers.

Let us start by defining what absolute convergence is. A series is called absolutely convergent if the series of absolute values of the terms is convergent,i.e.

$$\sum_{n=1}^{\infty} x_n \text{ is absolutely convergent if } \sum_{n=1}^{\infty} |x_n| \text{ is convergent}$$

Now the point of establishing absolute convergence is that, if a series is absolutely convergent than it is also convergent. To prove this let us consider a convergent series in the form of $\sum_{n=1}^{\infty} |x_n|$. We also should take note that $x_n \leq |x_n|$ & $0 \leq |x_n|$. From this we can construct the following inequalities:

$$0 \leq x_n + |x_n| \leq 2|x_n|$$

As $\sum_{n=1}^{\infty} |x_n|$ is convergent than so is $2 \sum_{n=1}^{\infty} |x_n|$. Now let us construct two new sequences $a_n = x_n + |x_n|$ and $b_n = 2|x_n|$. Now by comparing the sequences b_n and a_n we notice that $a_n < b_n$ for all $n \in \mathbb{N}$. From this we can conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} x_n + |x_n| \text{ is convergent.}$$

Now we note that

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (x_n + |x_n| - |x_n|) = \sum_{n=1}^{\infty} (x_n + |x_n|) - \sum_{n=1}^{\infty} |x_n|$$

and therefore the series $\sum_{n=1}^{\infty} x_n$ is convergent as it is the difference of two convergent series.

□

This result is of course of no surprise and using it we can easily deduce convergence of some of the alternating series, such as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots ,$$

which is of course convergent as we have already proved that the corresponding absolute value series is convergent. Now if a series isn't absolutely convergent, it can still be convergent. Such series is called conditionally convergent and as will be seen later, extra caution needs to be taken when dealing with such series.

Many of the series, which are interesting mathematically are alternating series. An alternating series is simply a series, where the terms alternate between positive and negative. To deal with such series Leibniz has come up with a special test, which is now goes by his name. The Leibniz test tell us if an alternating series is convergent or divergent. It states that for an alternating series of the positive sequence (a_n), which is written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots ,$$

to be convergent two conditions have to be met:

1. (a_n) decreases monotonically
2. $(a_n) \to 0$

To prove this test let us consider the partial sum of such series up to $2k$ terms to ensure that the last member is negative and so we get:

$$S_{2k} = a_1 - a_2 + a_3 - a_4 + \ldots + a_{2k-1} - a_{2k} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2k-1} - a_{2k}).$$

By bracketing the terms in such a way we ensure that each bracket is non-negative, because of the fact that (a_n) is a monotonically decreasing sequence. From this fact it can be easily deduced that the sequence (S_{2k}) is monotonically increasing. Now if we consider the partial sum up to and including the $(2k+1)$th term, we get:

$$S_{2k+1} = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots - a_{2k} + a_{2k+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots _(a_{2k} - a_{2k+1}).$$

Once again we bracket the terms, but this time it is easy to that the sequence (S_{2n+1}) is monotonic decreasing. Now next we have to take note of the fact that:

$$S_{2k+1} = S_{2k} + a_{2k+1} \geq S_{2k}$$

and so by combining what we have learned, we get

$$S_2 \leq S_4 \leq S_6 \leq \ldots \leq S_{2k} \leq S_{2k+1} \leq \ldots \leq S_5 \leq S_3 \leq S_1.$$

From this we deduce that (S_{2k}) is bounded above by S_1 and therefore has a limit L_t (say), due to the fact that (S_{2k}) is monotonic increasing. Similarly (S_{2k+1}) is bounded below by S_2 and being a monotonically decreasing sequence has a limit L_b (say).

Now if we let $k \to \infty$ in the equation $S_{2k+1} = S_{2k} + a_{2k+1}$ we get that $L_b = L_t + 0$ and therefore $L_b = L_t = L$, where L is the value of the series and so the series is convergent as it has a finite sum.

□

As said before this is a very powerful tool to investigate the convergence of alternating series. In particular one of the most famous series in mathematics can be proven to be convergent using this method. This series is the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

To prove that it is convergent, firstly we have to prove that $a_n \to 0$, which is easily done as

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

It is also obvious that the sequence $(1/n)$ is monotonically decreasing and so it passes the Leibniz test and is therefore convergent. Now finding the value of the series will require a bit more effort. Firstly let us return to the harmonic series and let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

Now define $\gamma_n = H_n - ln(n)$ and consider $x \in [k, k+1]$ for all $k \in \mathbb{N}$. It is obvious that the following inequality holds:

$$\frac{1}{k+1} \leq \frac{1}{x} \leq \frac{1}{k}.$$

Now if we integrate everything in the corresponding region we get

$$\int_k^{k+1} \frac{dx}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \int_k^{k+1} \frac{dx}{k} \therefore \frac{1}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \frac{1}{k}.$$

If we sum all the intervals from $k = 1$ to $k = n - 1$ we get that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} \leq \int_1^n \leq 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}.$$

which simplifies to

$$H_n - 1 \leq ln(n) - ln(1) \leq H_n - \frac{1}{n} \text{ and } \therefore -1 \leq ln(n) - H_n \leq -\frac{1}{n}$$

From this we deduce that

$$\frac{1}{n} \leq \gamma_n \leq 1$$

and therefore the sequence (γ_n) is bounded by 0 and 1. What is left, is for us to prove that the sequence is monotonic decreasing as that would mean that it is convergent. To do this we consider the difference between two consecutive terms, $\gamma_{n+1} - \gamma_n$. Because of how we defined this sequence we get

$$\gamma_{n+1} - \gamma_n = (H_n + \frac{1}{n+1} - ln(n+1)) - (H_n - ln(n)) = \frac{1}{n+1} + ln(n) - ln(n+1),$$

which if we use our result from earlier

$$\frac{1}{n+1} + ln(n) - ln(n+1) = \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x} \leq \frac{1}{n+1} - \frac{1}{n+1}$$

and therefore $\gamma_{n+1} - \gamma_n \leq 0$ thus proving (γ_n) is a monotonic decreasing sequence and therefore convergent with the limit being γ where $0 < \gamma < 1$. This limit is what is known in mathematics as the Euler-Mascheroni constant. But we only need the fact that it's value is between 0 and

1. Returning to our alternating harmonic series and denoting it's partial sum up to $2k$ by S_{2k} we can deduce the following:

$$\begin{aligned} S_{2k} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ... - \frac{1}{2k} = \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2k}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{2k}\right) = \\ &= H_{2k} - H_k = (\gamma_{2k} + ln(2k)) - (\gamma_k + ln(k)) = \\ &= \gamma_{2k} - \gamma_k + ln\left(\frac{2k}{k}\right) = \gamma_{2k} - \gamma_k + ln(2) \end{aligned}$$

Now if we let $k \to \infty$ we have both $\gamma_{2k} \to \gamma$ and $\gamma_k \to \gamma$, therefore canceling each other out and all that we are left is $ln(2)$. Therefore we found the value of the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = ln(2).$$

There is a very interesting property of series, which are conditionally convergent. This property is due to what is called the Riemann series theorem. The property is that the terms of a conditionally convergent series can be rearranged so that the series takes the value of any real number. Or even more strangely the terms can be rearranged in such a way, so that the series diverges to $+\infty$ or $-\infty$[6]. This extremely unexpected and mathematically astonishing result is quite tricky to prove and rather than trying to prove the result of one of the greatest minds of 19th century, we will just show an example, which confirms this theory.

Let us go back to the alternating harmonic series and let it's sum be S and it's partial sum S_n. Now consider a rearrangement of this series, which converges to R, given by

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + ...$$

and let it's partial sum to n be denoted by R_n. Also label the partial sum of the harmonic series up to n as H_n. For us to prove that $R \neq S$ we need to prepare by considering the partial sum R_{3n}. So

$$\begin{aligned} R_{3n} &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + ... + \frac{1}{4n_3} + \frac{1}{4n-1} - \frac{1}{2n} = \\ &= 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{4n-1} - \left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2n}\right) = \\ &= 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{4n-1} - \frac{1}{2}H_n = \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{4n-1} + \frac{1}{4n} - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{4n}\right) - \frac{1}{2}H_n \\ &= H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n. \end{aligned}$$

Next we have to take note that:

$$S_{2n} = H_{2n} - H_n$$

as was proven earlier, when we were proving the sum of the alternating harmonic series. Now using this info we get that

$$R_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n = S_{4n} + \frac{1}{2}S_{2n}.$$

Finally if we let $n \to \infty$

$$R_{3n} \to S + \frac{1}{2}S$$

$$\therefore R = \frac{3}{2}S \ ,$$

which is an astonishing result as you could say we found another value for the alternating harmonic series. (Truly it isn't another value of the series, it can only be associated with the rearranged series).

This concludes this essay, but the topic of analysis continues way beyond this point. This topic also contains many modern unsolved mathematical problems. Perhaps the most famous one of them being the Riemann Hypothesis concerning an extremely important function called the Riemann zeta function, and most of the examples in this essay are values of the zeta function at specific points, including the natural number series, but to understand that a far more advanced level of mathematics is needed. Overall throughout the essay we have seen how much the mathematics of infinite series has advanced from Zeno to Riemann.

References

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[7] An extremely helpful resource, which helped me throughout the essay: John.M owie *Real Analysis*, Springer 2001